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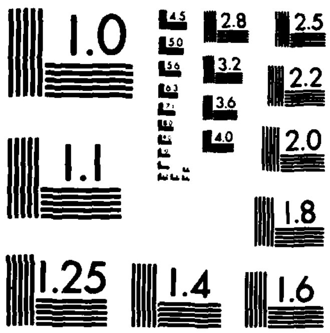

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Research Report CCS 456

HOMOCORES, CORES AND OPERATIONAL INEFFICIENCY IN SUPERADDITIVE N-PERSON GAMES

by

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ABSTRACT

Defects in the core as a solution to n-person superadditive characteristic function games are examined and the process of achieving a "reasonable" core by changing the value of the grand coalition is studied. A new unique solution concept, the "homocore," is proposed based on the "homomollifier" notion interpreted as the result of implicit bargaining and weighted averages of coalitional worth. Thereby unreasonable cores are excluded but a core-like dominance property on the average is maintained. It also yields a measure of "operational efficiency" where the game may be interpreted as an economic situation of decreasing marginal productivity.

KEY WORDS

Core

Homocore

Homomollifier

Operational Efficiency

Really Essential Games

HOMOCORES, CORES AND OPERATIONAL INEFFICIENCY IN SUPERADDITIVE N-PERSON GAMES

by

A. Charnes and B. Golany

1. Introduction

The core, as a solution concept for an n-person characteristic function game is often criticized for three major deficiencies. It can be empty, thus leaving us with no solution. It may consist of an infinite number of imputations, each having equal a priori claim to be the final agreement among the players. It can also include only one imputation, but one which is not plausible in reality. (See the BL^2 game in Charnes, Littlechild and Sorensen [1973]). Other important criticisms and suggestions of directions for modifications were made in L. Johanssen's working papers at The Econometric Institute of The University of Oslo in 1981 and in Aumann [1981].

Yet we know that with respect to the concept of dominance, the core has the important advantage of stability. In order to move toward such stability, Charnes and Kortanek [1967] showed that any empty core game can be turned into one with a core by changing at most one value of the characteristic function, (e.g. the value of the grand coalition v(N)).

However, as we show here, it is not always possible to find an amount which, added to the v(N) of an empty core game, yields a single member core. On the other hand, the "homocore", shortly to be defined, which is based on a minimal increase in v(N) (and for which one needs to consider only one level of the coalitional inequalities which define the core) always exists and is unique.

Besides dominance stability, other properties of solutions are also important. Thus Charnes/Rousseau/Seiford [1977] developed the notion of an incremental propensity to disrupt, and the "homomollifier" as a conclusion to an implicit process of bargaining. By definition, the characteristic function (designated "coalitional worth" by Aumann [1981]) gives the value which a coalition can achieve regardless of the actions taken by the other players. It does not directly measure "anti-coalitional strength", namely--the power to block formation of other coalitions by preempting members. To bring this strength into consideration in characteristic function form, we use the homomollifier process as a "mapping" of anti-coalitional strength into changes of coalitional worth value.

The homomollifier of any really essential superadditive game is essential with empty core. (By "really" we mean some (n-1)-person subgame is essential.) Thus, by converting the original game into its homomollifier, we bring all really essential games (with core, without, or with a unique core) to the same starting point. Then we raise the value of v(N) until *all* the cardinality levels of the coalitional inequalities which define the core are satisfied. Then we obtain a unique imputation, based on the average value for each player over all the coalitions in which he participates at the level which yields a core, which is projected downward to sum to the original v(N). This imputation we call the "homocore." For not really essential games, the homomollifier is inessential and we take as the homocore its unique imputation (which is the core).

An interesting yield of this new concept is brought out in relating game-like situations to economic situations of decreasing marginal productivity in the second part of this paper. Employing this analogy we show that this concept gives a reasonable measure of the relative "operational efficiency" of different production units.

But one needs also to consider whether or not a solution notion corresponds to what people might select in practice. Thus Heaney [1978] reviewed attempts to find an acceptable solution concept for real large project situations which required allocation of costs to participants. Allocations in the core were indeed attractive to these real participants. Selton/Schuster [1968] also studied psychological criteria and effects. But in an intensive experimental comparison of core, Shapley value, nucleolus and other notions, Michener, Macheel, Depies and Bowen [1983] showed that use of the homomollifier to reshape the characteristic function gave easily the best correspondence to human choices. We show here that the homocore does substantially better than the best of these on this experimental data.

2. Unique Core Imputations

Let (N,v) be a characteristic function game where $N = \{1,2,\ldots,n\}$ is the set of players and v is a non-negative function defined on all subsets of N with $v(\emptyset) = 0$. Let x be a payoff vector with elements x_i, $i = 1,2,\ldots,n$, and let S be any subset of N with $|S|$ as the cardinality of the set S.

Theorem 2.1: Any inessential game has a unique imputation in the core:

$$x(i) = v(i) \ , \ \forall i \in N.$$

Proof: An inessential game is defined by $\sum_{i \in N} v(i) = v(N)$. According to the definition of the core, x should satisfy:

$$x(S) \geqslant v(S) \quad \forall S \subset N \quad \text{and} \quad \sum_{i \in N} x_i = v(N).$$

Thus $x(i) \geqslant v(i)$, $\forall i \in N$. Supose that for a certain $j \in N$, $x(j) > v(j)$. Then $\sum_{i \in N} x(i) > \sum_{i \in N} v(i) = v(N)$. This cannot happen since $\sum_{i \in N} x(i) = v(N)$. Hence the only possible imputation is $x(i) = v(i)$, $\forall i \in N$.

Q.E.D.

We define the homocore as this unique imputation wherever the homomollifier is inessential, i.e., iff all (n-1)-person subgames of the original game are inessential (see [13]).

Theorem 2.2: A superadditive essential game has a unique imputation in the core if

$$(1)\quad v(N) = \frac{1}{n-1}\sum_{|S|=n-1} v(S) = \max_{k<n}\left(\frac{1}{\binom{n-1}{|S|-1}}\right)\sum_{|S|=k} v(S)$$

$$(2)\quad v(S) \leqslant \frac{1}{n-1}\left[|S|\sum_{i\notin S} v\{N-i\} - (n-1-|S|)\sum_{i\in S} v\{N-i\}\right],$$

$$\forall |S| < n-1$$

Proof: Here we need to construct the system of inequalities which define the core. Notice that:

(i) The system consists of n levels: $x(S) \geqslant v(S)$, $|S| = 1,2,\ldots,n$.

(ii) The number of inequalities in the k^{th} level is $\binom{n}{k}$.

(iii) The number of times each x_i $(i=1,2,\ldots,n)$ appears in all the inequalities of the k^{th} level is $\binom{n-1}{k-1}$.

The conditions for x to be in the core are:

$$(1)\quad \begin{array}{lll}
\left.\begin{array}{l}
x_1 \geqslant v_1 \\
x_2 \geqslant v_2 \\
\quad\vdots \\
x_n \geqslant v_n
\end{array}\right\} & \text{1st level} \\
\left.\begin{array}{l}
x_1+x_2 \geqslant v_{1,2} \\
\quad\vdots \\
x_{n-1}+x_n \geqslant v_{n-1,n}
\end{array}\right\} & \text{2nd level} \\
\quad\vdots \\
\left.\begin{array}{l}
x_1+x_2+\ldots+x_{n-2} \geqslant v_{1,2,3,\ldots,n-2} \\
\quad\vdots \\
x_3+x_4+\ldots+x_{n-1}+x_n \geqslant v_{3,4,\ldots,n}
\end{array}\right\} & n-2^{th}\text{ level} \\
\left.\begin{array}{l}
x_1+x_2+\ldots+x_{n-1} \geqslant v_{1,2,3,\ldots,n-1} \\
\quad\vdots \\
x_2+x_3+\ldots+x_{n-1}+x_n \geqslant v_{2,3,\ldots,n}
\end{array}\right\} & n-1^{th}\text{ level} \\
\left.x_1+x_2+\ldots+x_n = v(N)\right\} & n^{th}\text{ level.}
\end{array}$$

I. Summing the n inequalities in the $(n-1)^{th}$ level (in which each x_i appears n-1 times) we get:

$$(n-1) \cdot x(N) \geqslant \sum_{|S|=n-1} v(S).$$ Now substituting $x(N) = v(N)$ from the n^{th} level we have:

$$v(N) \geqslant \frac{1}{n-1} \sum_{|S|=n-1} v(S)$$

II. Doing the same for the $n-2^{th}$ level gives: $v(N) \geqslant \frac{2}{(n-1)(n-2)} \sum_{|S|=n-2} v(S)$

III. Doing the same for the k^{th} level gives: $v(N) \geqslant \frac{1}{\binom{n-1}{k-1}} \sum_{|S|=k} v(S)$

To satisfy the new set of inequalities (III above), it is sufficient to satisfy the single inequality, $v(N) \geqslant \max_{|S|<n} \frac{1}{\binom{n-1}{|S|-1}} \sum_{|S|} v(S)$. By condition (1) this occurs for $|S|=n-1$. Before we reduce the original system of 2^n-1 inequalities to the subset in which $|S|=n-1$ plus the last equation (a total of n+1 inequalities), we have to make sure that no exceptional inequality in the levels $|S| < n-1$ will cause a contradiction. Therefore we require condition (2) to hold for all $|S|<n-1$. Since our imputation in the core is to be $x_i = v(N) - v\{N-i\}$ we must have:

$$v(S) \leqslant \sum_{i\in S} \left[v(N) - v\{N-i\}\right] = |S|v(N) - \sum_{i\in S} v\{N-i\} = \frac{|S|}{n-1} \sum_{i=1}^{n} v\{N-i\} - \sum_{i\in S} v\{N-i\}$$

$$= \left[\frac{|S|}{n-1} - 1\right] \sum_{i\in S} v\{N-i\} + \frac{|S|}{n-1} \sum_{i\notin S} v\{N-i\} = \frac{1}{n-1} \left[|S| \sum_{i\notin S} v\{N-i\} - (n-1-|S|) \sum_{i\in S} v\{N-i\}\right].$$

Thus condition (2) guarantees that all the constraints in [1] for $|S| < n-1$ are redundant to the question of determining the core. Now we are left with precisely n+1 inequalities:

$$
(2)\qquad
\begin{array}{l}
x_1+x_2+\ldots \qquad \ldots+x_{n-1} \geq v_{1,2,\ldots,n-1} \\
x_1+x_2+\ldots \qquad \ldots+x_{n-2} \quad +x_n \geq v_{1,2,\ldots,n-2,n} \\
\vdots \\
\quad x_2+x_3+\ldots \qquad \ldots+x_n \geq v_{2,3,\ldots,n} \\
x_1+x_2+\ldots \qquad \ldots+x_n = v(N) = \frac{1}{n-1} \sum_{|S|=n-1} v(S)
\end{array}
$$

Now suppose for a solution $(x_1,\ldots,x_n)$ that in one of the inequalities above the left side is strictly greater than the right. Then, adding the first n inequalities we get the contradiction $\binom{n-1}{n-2}\, x(N) > \sum_{|S|=n-1} v(S)$, since $x(N) = v(N) = \frac{1}{n-1} \sum_{|S|=n-1} v(S)$. Hence, what we really have are n+1 equations. Subtracting each of the first n from the last one, we get uniquely

$$
(3)\qquad
\begin{array}{l}
x_n = v(N) - v_{1,2,\ldots,n-1} \\
x_{n-1} = v(N) - v_{1,2,\ldots,n-2,n} \\
\quad\ddots \\
x_1 = v(N) - v_{2,3,\ldots,n}
\end{array}
$$

the imputation we employed in condition (2).

Each of the elements on the right side is non-negative due to superadditivity. Also, by superadditivity, $x_i \geqslant v(i)$. Thus we have a unique, feasible solution.[1]

Q.E.D.

2.1 3-Person Empty Core Games

It is not difficult to show that any superadditive 3-person, empty core game can be represented in 0-1 normalized form as follows:

$N = \{1,2,3\}$

$v(\emptyset) = v(i) = 0$, $i=1,2,3$

$v(12) = a$, $v(13) = b$, $v(23) = c$, where $a,b,c \in [0,1]$, $a+b+c > 2$.

$v(N) = 1$.

Lemma 1: For any (N,v) game as defined above, the maximum over $k < n$ in condition (1) of Theorem 2.2 occurs for $k = 2$.

Proof: v is a real, non-negative function and

$$\frac{1}{n-1}\sum_{|S|=2} v(S) \geqslant 0 = \sum_{i=1}^{3} v\{i\}.$$

Q.E.D.

Lemma 2: For any (N,v) game as defined above, condition (2) of Theorem 2.2 always holds.

Proof: Since $n = 3$, the only sub-coalition level we need to check is $|S|=1 < n-1$. For this level we substitute in condition (2):

---

[1]Our homoco[illegible] solution under these hypotheses on $v'(N)$ will be:

$$x(i) = [v'(N) - v(N-i)] \cdot \frac{v(N)}{v'(N)} , \quad i=1,2,\dots,n ,$$

where $v'(N) = \frac{1}{n-1}\sum_{|S|=n-1} v(S)$. It is the solution to system (3) projected by $v(N)/v'(N)$ to satisfy $x(N) = v(N)$.

$$v(S) = 0 \leqslant \frac{1}{2}\left[(1)\cdot\sum_{i\notin S} v(N-i) - (3-1-1)\sum_{i\in S} v(N-1)\right] = \begin{cases} \frac{a+b-c}{2}, & S=\{1\} \\ \frac{a+c-b}{2}, & S=\{2\} \\ \frac{b+c-a}{2}, & S=\{3\}. \end{cases}$$

Without loss of generality, we check $S = \{1\}$. Employing the empty core property $a+b+c > 2$ we have: $a+b+c > 2 \Rightarrow a+b-c > 2 - 2c \Rightarrow \frac{a+b-c}{2} > 1 - c$. But $1 - c \geqslant 0$ because $c \in [0,1]$, so we have $\frac{a+b-c}{2} > 0$. By changing the names a, b, c, we also have checked $S = \{2\}$ and $S = \{3\}$.

Q.E.D.

Hence we know that by "raising" v(N) to equal $\frac{a+b+c}{2}$ we satisfy the first part of condition (1) in Theorem 2.2. Since condition (2) always holds we have a unique solution based on Theorem 2.2:

$$x_1 = \frac{a+b-c}{2} \quad ; \quad x_2 = \frac{a+c-b}{2} \quad ; \quad x_3 = \frac{b+c-a}{2} \quad .$$

This <u>is</u> an imputation for the <u>new</u> v(N) since $a+b+c > 2$. Next, to get an imputation from this,valid for the <u>old</u> v(N), we have to "reduce" x(N). This is done by multiplying each above x_i by $\frac{2}{a+b+c}$.

Thereby, we have for any (N,v) game as defined above, the unique solution:

$$x_1 = 1 - \frac{2c}{a+b+c} \quad ; \quad x_2 = 1 - \frac{2b}{a+b+c} \quad ; \quad x_3 = 1 - \frac{2a}{a+b+c} \quad .$$

Going back to the $n > 3$ situation, if $\frac{1}{n-1} \sum_{|S|=n-1} v(S) < \max_{|S|<n} \frac{1}{\binom{n-1}{|S|-1}} \sum_{|S|} v(S)$, then in general there is no minimal value of $v(N)$ which yields a single member core. For, if the maximum occurs at the k^{th} level, then in the last stage of the previous proof we would have $\binom{n}{k}$ equations in n unknowns. For $1 < k < n-1$, $\binom{n}{k} > n$, which yields a rectangular matrix. This will have a solution only if the additional conditions (preventing contradiction at the $n\text{-}k^{th}$ level of the original set, and yielding $\binom{n}{k}$ -n equations to linear independent of the other n equations) are satisfied--which is not true for the general case.

<u>Theorem 2.3</u>: The homomollifier of a superadditive really essential game has an empty core.

<u>Proof</u>: The homomollifier is a really essential superadditive constant sum game. (See [13] and [4] Theorem 3.7.) Consider the system which defines the core in the form:

$$
\begin{aligned}
x(S_1) &\geq w(S_1) \\
&\vdots \\
x(N-S_1) &\geq w(N-S_1) \\
&\vdots \\
x(N) &= w(N)
\end{aligned}
$$

If for any S_1, $x(S_1) > w(S_1)$, we have a contradiction when we add since

$$x(S_1) + x(N-S_1) > w(S_1) + w(N-S_1) = w(N) ,$$

i.e. $x(N) > w(N)$.

Thus we have only $x(S) = w(S)$, $\forall S \subseteq N$. But this yields an inessential game, a contradiction. Therefore, in an essential game, the condition above cannot hold and we have an empty core.

Q.E.D.

Since we have decided to use the homomollifier as a representative of an implicit bargaining process, we shall always have an empty core game to start with. For this kind of game, although we know ([2] proposition 6) that by raising $v(N)$ we can have a non-empty core game, the core will not necessarily be a unique imputation (Corollary 2.1). To achieve uniqueness while coming close to (or equaling) the downward projection of a core solution, we propose the following "homocore" solution.

3. The Homocore Solution

The homocore is defined by:

$$\boxed{x(i) = \frac{n-1}{n-k}\left[\sum_{\substack{|S|=k \\ i\in S}} w(S) - \frac{k-1}{n-1}\sum_{|S|=k} w(S)\right]\frac{w(N)}{\sum\limits_{|S|=k} w(S)} , \quad i = 1,2,\ldots,n}$$

where w is the homomollifier of the original game v, and k satisfies:

$$\max_{|S|<n} \frac{1}{\binom{n-1}{|S|-1}} \sum_{|S|} w(S) = \frac{1}{\binom{n-1}{k-1}} \sum_{|S|=k} w(S) .$$

Here we first changed w(N) to equal $\max\limits_{|S|<n} \dfrac{1}{\binom{n-1}{|S|-1}} \sum\limits_{|S|} w(S)$. Then instead of system (2) of Theorem 2.2 we have:

$$\begin{array}{llllll}
x_1+x_2+\dots & & +x_k & & & \geqslant w_{1,2,\dots,k-1,k}\\
x_1+x_2+\dots & +x_{k-1} & & +x_{k+1} & & \geqslant w_{1,2,\dots,k-1,k+1}\\
\vdots & & & & \ddots & \vdots\\
x_1+x_2+\dots & +x_{k-1} & & & +x_n & \geqslant w_{1,2,\dots,k-1,n}\\
x_1+x_2+\dots & +x_{k-2} & +x_k & +x_{k+1} & & \geqslant w_{1,2,\dots,k-2,k,k+1}\\
x_1+x_2+\dots & +x_{k-2} & +x_k & +x_{k+2} & & \geqslant w_{1,2,\dots,k-2,k,k+2}\\
\vdots & & & & \ddots & \vdots\\
x_1+x_2+\dots & +x_{k-2} & +x_k & & +x_n & \geqslant w_{1,2,\dots,k-2,k,n}\\
 & & & & & \vdots\\
 & x_{n-k+1}\dots & & & \dots+x_n & \geqslant w_{n-k+1,\dots,n}\\
x_1+x_2+\dots & & \dots+x_k\dots & & \dots+x_n & = \dfrac{1}{\binom{n-1}{k-1}} \sum\limits_{|S|=k} w(S)
\end{array}$$

For the same reason explained in Theorem 2.2, all the inequalities are actually equalities. Now we sum for each x(i) all the rows in which x(i) occurs. This yields the following set of n equations:

$$\binom{n-1}{k-1} x(i) + \binom{n-2}{k-2} x(N-i) = \sum_{\substack{|S|=k\\ i\in S}} w(S) \quad , \quad i=1,\dots,n$$

Thus

$$\left[\binom{n-1}{k-1} - \binom{n-2}{k-2}\right] x(i) + \binom{n-2}{k-2} x(N) = \sum_{\substack{|S|=k\\ i\in S}} w(S)$$

and

$$\binom{n-2}{k-1} x(i) + \frac{\binom{n-2}{k-2}}{\binom{n-1}{k-1}} \sum_{|S|=k} w(S) = \sum_{\substack{|S|=k\\ i\in S}} w(S) \qquad \text{(substitute } x(N) \text{ from the last equation)}$$

$$\binom{n-2}{k-1} x(i) = \sum_{\substack{|S|=k \\ i\in S}} w(S) - \frac{k-1}{n-1} \sum_{|S|=k} w(S)$$

so that

$$x(i) = \frac{1}{\binom{n-2}{k-1}} \left[\sum_{\substack{|S|=k \\ i\in S}} w(S) - \frac{k-1}{n-1} \sum_{|S|=k} w(S) \right]$$

Multiplying this result by $\dfrac{\binom{n-1}{k-1} w(N)}{\sum_{|S|=k} w(S)}$ to project our imputation down in value to sum to the original $w(N)$, we get:

$$x(i) = \frac{\binom{n-1}{k-1}}{\binom{n-2}{k-1}} w(N) \left[\frac{\sum_{\substack{|S|=k \\ i\in S}} w(S)}{\sum_{|S|=k} w(S)} - \frac{k-1}{n-1} \right] =$$

$$= w(N) \left[\frac{(n-1) \sum_{\substack{|S|=k \\ i\in S}} w(S)}{(n-k) \sum_{|S|=k} w(S)} - \frac{k-1}{n-k} \right]$$

<u>Theorem 3.1</u>: The homocore solution satisfies:

(1) $x(N) = v(N)$

(2) is uniquely determined.

<u>Proof</u>: (1) $x(i) = w(N) \left[\dfrac{(n-1) \sum_{\substack{|S|=k \\ i\in S}} w(S)}{(n-k) \sum_{|S|=k} w(S)} - \dfrac{k-1}{n-k} \right]$

$$x(N) = \sum_{i=1}^{n} x(i) = w(N) \sum_{i=1}^{n} \left[\frac{(n-1) \sum_{\substack{|S|=k \\ i\in S}} w(S)}{(n-k) \sum_{|S|=k} w(S)} - \frac{k-1}{n-k} \right] =$$

$$= w(N)\left[\frac{n-1}{n-k}\,\frac{\sum_{i=1}^{n}\sum_{\substack{|S|=k\\ i\in S}} w(S)}{\sum_{|S|=k} w(S)} - \frac{n(k-1)}{n-k}\right] =$$

$$= w(N)\left[\frac{n-1}{n-k}\,\frac{k\cdot\sum_{|S|=k} w(S)}{\sum_{|S|=k} w(S)} - \frac{n(k-1)}{n-k}\right] = w(N)\left[\frac{(n-1)k-n(k-1)}{n-k}\right] =$$

$$= w(N) = v(N)$$

(2) From the construction of x, we have a set of n equations in n unknowns with only one unknown in each equation.

Q.E.D.

Theorem 3.2: In the special case for which $k = n-1$, the homocore is the solution obtained in Theorem 2.2.

Proof: For $k = n-1$ we get:

$$x(i) = w(N)\left[\frac{(n-1)\sum_{\substack{|S|=n-1\\ i\in S}} w(S)}{\sum_{|S|=n-1} w(S)} - (n-2)\right] =$$

$$= w(N)\left[\frac{\sum_{\substack{|S|=n-1\\ i\in S}} w(S)}{w'(N)} - \frac{\frac{n-2}{n-1}\sum_{|S|=n-1} w(S)}{w'(N)}\right] \qquad \text{where } w'(N) = \frac{1}{n-1}\sum_{|S|=n-1} w(S)$$

$$= \frac{w(N)}{w'(N)} \left[\frac{n\left(\sum\limits_{\substack{|S|=n-1\\ i\in S}} w(S) - \sum\limits_{|S|=n-1} w(S)\right) + 2\sum\limits_{|S|=n-1} w(S) - \sum\limits_{\substack{|S|=n-1\\ i\in S}} w(S)}{n-1} \right]$$

$$= \frac{w(N)}{w'(N)} \left[\frac{-nW(N-i) + w(N-i) + \sum\limits_{|S|=n-1} w(S)}{n-1} \right] =$$

$$= \frac{w(N)}{w'(N)} \left[\frac{1}{n-1} \sum_{|S|=n-1} w(S) - \frac{n-1}{n-1} w(N-i) \right] = \frac{w(N)}{w'(N)} \left[w'(N) - w(N-i) \right].$$

Q.E.D.

We next consider some examples.

<u>Examples 3.1</u>

3.1.1, The BL^2 game: $v(1) = v(2) = v(3) = v(23) = 0$; $v(12) = v(13) = v(123) = 1$

The homomollifier is: $w(1) = \frac{1}{3}$; $w(2) = w(3) = 0$; $w(12) = w(13) = w(123) = 1$; $w(23) = \frac{2}{3}$.

The homocore is: $x_1 = \frac{2}{1} [(1+1) - \frac{1}{2}(1+1+\frac{2}{3})] \frac{1}{(1+1+\frac{2}{3})} = \frac{1}{2}$ (k=2)

$x_2 = x_3 = \frac{2}{1} [(1+\frac{2}{3}) - \frac{1}{2}(1+1+\frac{2}{3})] \frac{1}{(1+1+\frac{2}{3})} = \frac{1}{4}$; $(x(N)=1)$.

This result actually reflects a situation in which players 2,3 join to block player 1 from getting everything (as given by the original core). Therefore, the homomollifier evidently represented an implicit process in which players 2,3 created a union which had the same bargaining power of player 1, thus causing a fair division of the total $v(N)$.

3.1.2, An empty core game: $v(1) = v(2) = v(3) = 0$; $v(12) = v(123) = 1$; $v(13) = 5/6$; $v(23) = 4/6$. The homomollifier is: $w(1) = \frac{2}{18}$; $w(2) = \frac{1}{18}$; $w(3) = 0$; $w(12) = w(123) = 1$; $w(13) = \frac{17}{18}$; $w(23) = \frac{16}{18}$

The homocore is: $k = 2$; $\sum_{|S|=2} w(S) = \frac{51}{18}$; $\frac{n-1}{n-k} = 2$; $\frac{k-1}{n-1} = \frac{1}{2}$;

$$x(1) = \frac{19}{51} \; ; \; x(2) = \frac{17}{51} \; ; \; x(3) = \frac{15}{51} \; ; \; x(N) = 1 = v(N).$$

Note that in Examples 3.1 our games satisfied the condition

$\frac{1}{n-1} \sum_{|S|=n-1} v(S) = \max_{|S|<n} \left(\frac{\frac{1}{n-1}}{|S|-1}\right) \sum_{|S|} v(S)$. The next example shows a more complex situation.

Example 3.2

An empty core, superadditive and essential 5-person game:

$v(i) = 0 \quad , \quad \forall i=1,2,\ldots,5$

$v(ij) = 0 \quad , \quad \forall i,j=1,2,\ldots,5 \; , \; i \neq j$

$v(12j) = 3.5 \; , \; \forall j=3,4,5$

$v(34j) = 3 \; , \quad \forall j=1,2,5$

$v(135) = v(145) = v(235) = v(245) = 4$

$v(ijkl) = 4.5 \; , \; \forall i,j,k,l=1,2,\ldots,5 \; , \; i \neq j \neq k \neq l$

$v(12345) = 5$.

Here $\max_{|S|<n} \left(\frac{\frac{1}{n-1}}{|S|-1}\right) \sum_{|S|} v(S) = \Big|_{|S|=3} 5.91666$. The homomollifier is:

$w(i) = 0.1 \; , \; \forall i=1,2,\ldots,5$

$w(12) = w(15) = w(25) = 0.8$

$w(13) = w(14) = w(23) = w(24) = 0.4$

$w(34) = w(35) = w(45) = 0.6$

$w(12j) = 4.4 \; , \; \forall j=3,4,5$

$w(34j) = 4.2 \; , \; \forall j=1,2,5$

$w(135) = w(145) = w(235) = w(245) = 4.6$

$w(ijkl) = 4.9 \; , \; \forall i,j,k,l=1,2,\ldots,5 \; , \; i \neq j \neq k \neq l$

$w(12345) = 5$

Again $\max\limits_{|S|<n} \frac{1}{\binom{n-1}{|S|-1}} \sum\limits_{|S|} w(S) = \Big|_{|S|=3} 7.366$. Therefore $k = 3$, $\frac{n-1}{n-k} = 2$,

$\frac{k-1}{n-1} = \frac{1}{2}$ and so:

$x_1 = 1.018$, $x_2 = 1.018$, $x_3 = 0.9276$, $x_4 = 0.9276$, $x_5 = 1.1086$, $x(N) = 5$.

The example above has no minimal value which added to $w(N)$ yields a unique solution.

Theorem 3.3: For any essential superadditive empty core game w, the solution to the following extremal problem gives the minimal value $w'(N)$ for which the game has a non-empty core:

$$\begin{aligned} &\min \quad x(N) \\ &\text{s.t.} \quad x(S) \geq g(S) \ , \ \forall\ |S| < n \end{aligned}$$

Proof: The dual to the problem above is:

$$\begin{aligned} &\min \quad w^T G \\ &\text{s.t.} \quad w^T Y = e^T \\ &\qquad\ \ w^T \geq 0 \end{aligned}$$

where $G^T = \{g(1),g(2),\ldots,g(n),g(12),\ldots\}$,

and Y is a matrix of entries 1,0, whose 1 entries in each row correspond to the elements of S. It is clear that the dual is consistent (take $w^T = \{\underbrace{1,1,\ldots,1}_{n},\underbrace{0,0,\ldots,0}_{(2^n-2)-n}\}$ and remember that Y has the identity matrix $I_{n\times n}$ as its first n rows). It is also clear that the primal is consistent since there is no upper bound on $x(N)$. Hence, by the linear programming duality states theorem, both have the same optimal solution value: $x^*(N) = w^{*T}G$. Now if we substitute $g'(N) = x^*(N)$ we have a new game $G' = \begin{Bmatrix} g(S), & |S|<n \\ g'(S), & |S|=n \end{Bmatrix}$ in which the conditions for the existence of the core hold.

Q.E.D.

Example 3.3

Consider the game in 3.2. As shown before $\max_{|S|<n} \frac{1}{\binom{n-1}{|S|-1}} \sum_{|S|} v(S) = \Big|_{|S|=3} 5.91666.$

This creates a 11-row 5-column matrix which is not consistent. Using the simplex method (applied on the formulation in Theorem 3.2) gives a minimal value $v'(N) = 6.1666$ (for which $x_1 = x_2 = 1.333$; $x_3 = x_4 = 0.8333$; $x_5 = 1.8333$ is an imputation). This is the minimum which guarantees a non-empty core. Applying the same method to the homomollifier of the game gives $w'(N) = 7.4666$ with a projected imputation: $x_1 = x_2 = 1.026$; $x_3 = x_4 = 0.893$; $x_5 = 1.16$ ($x(N) = 5$). This solution is rather close to our homocore solution found in Ex. 3.2. but the problem with it is that it is not necessarily a unique one. However, we shall later employ $w'(N) = 7.466$ in our efficiency measure.

4. A Measure of Operational Inefficiency

Consider a superadditive characteristic function game as describing a situation in which we have a pool of workers to perform some production project. There are n workers in the pool, each with various skills and experience. Their efforts in coalitions are compensated in amounts as given by the characteristic function. Different groups (coalitions) of workers can achieve different norms of production, and assuming correspondence between the produced quantities and payments, we assume the higher values of $v(S)$ are for higher production. As we know, in reality there are no perfect production situations. We propose to describe operational efficiency in this analogy in terms of the incremental value added to the value of the grand coalition which will achieve a non-empty core.

We define $E_{N,v}$, our measure of operational efficiency by

$$E_{N,v} \triangleq 1 - \frac{w'(N) - w(N)}{w'(N)} = \frac{w(N)}{w'(N)}$$

Let us consider some examples.

Example 4.1:

The BL^2 game has $v(1) = v(2) = v(3) = v(23) = 0$, $v(12) = v(13) = v(123) = 1$.

The homomollifier gives $w(N) = 1$, $w'(N) = 4/3$ hence:

$$E_{N,v} = \frac{1}{4/3} = 0.75$$

This corresponds to the fact that in the BL^2 situation there is a redundancy of one little man. His marginal contribution, when he joins a team of the big man and the other little man, is zero!

Example 4.2:

Modify the BL^2 game to $v(1) = v(2) = v(3) = v(23) = 0$, $v(12) = v(13) = 1$, $v(123) = 2$.

This is clearly a more efficient situation since the output of the total team is higher. But note that while the relative improvement over that of the first little man is infinite ($v(12)/v(1) = \infty$), that over the second is a relative improvement of 1 ($v(123)/v(12) = 1$. This means that although it is a better situation, it is not a perfect one. Let us check the value of $E_{N,v}$ for this game:

Here $w(1) = 2/3$, $w(2) = w(3) = 1/3$, $w(12) = w(13) = 5/3$, $w(23) = 6/3$, $w(123) = 2$, $w'(123) = 7/3$, so $E_{3,v} = \frac{2}{7/3} = 6/7 = 0.857$.

As expected, the efficiency measure here is higher than in the BL^2 game.

Example 4.3:

$$v(S) = 1 \ , \ \forall|S| = 8,9,10 \ ; \ v(S) = 0 \ , \ \forall|S| = 0,1,\ldots,7$$

It is obvious that we have a redundancy of two workers in this situation. Using the same method we get:

$$w(10) = 1 \ , \ w'(10) = 10/8 \ ; \ E_{10,v} = \frac{1}{10/8} = 0.8$$

Here we see a new aspect of the efficiency measure. It can help in comparing situations with different numbers of workers. As will be shown in the next example, the number of redundant workers is an indicator of the level of inefficiency.

Example 4.4

$$v(S) = 1 \ , \ \forall|S| = 6,7,8,9,10 \ ; \ v(S) = 0 \ , \ \forall|S| = 0,1,\ldots,5$$

Here we have four redundant workers compared to only two in the previous example. Here $w(10) = 1$, $w'(10) = 10/6$, hence $E_{10,v} = \frac{1}{10/6} = 0.6$.

Changes in efficiency may be anticipated only when the outputs $(v(N),v(S))$, or the inputs (number of workers for the same mission), are changed. In the next example we reduce the number of workers for the same task as in Example 4.4.

Example 4.5

$$v(S) = 1 \ , \ |S| = 9,8,7,6 \ ; \ v(S) = 0 \ , \ \forall|S| = 0,1,\ldots,5$$

Here $w(N) = 1$; $w'(N) = 1.5$, $E_{N,v} = \frac{1}{1.5} = 0.66$. Thus, the reduction gives an improvement in the efficiency measure.

Theorem 4.1: For any superadditive and essential game v, the measure of operational efficiency satisfies $0.5 < E_{N,v} < 1$.

Proof: We choose $w'(N) = \frac{1}{\binom{n-1}{k-1}} \sum_{|S|=k} w(S)$ (see definition in section 3).

The constant sum property gives $w(N) = w(S) + w(N-S)$, $\forall S \subseteq N$

Consider first $E_{N,v} \leqslant 1$. For all S, $w(N) = w(S) + w(N-S)$. In particular for $S = k$,

$$\frac{1}{\binom{n-1}{k-1}} \sum_{|S|=k} w(N) = \frac{1}{\binom{n-1}{k-1}} \sum_{|S|=k} w(S) + \frac{1}{\binom{n-1}{k-1}} \sum_{|S|=k} w(N-S)$$

So,

$$\frac{n}{k} w(N) = \frac{\binom{n}{k}}{\binom{n-1}{k-1}} w(N) = w'(N) + \frac{1}{\binom{n-1}{k-1}} \sum_{|S|=k} w(N-S) \leqslant 2w'(N)$$

There is only one case in which the above inequality can be an equality and this occurs when $k = \frac{n}{2}$. Then we have $\frac{n}{n/2} w(N) \leqslant 2w'(N)$, implying $E_{N,v} \leqslant 1$. In all other cases we have $w'(N) > \frac{1}{\binom{n-1}{k-1}} \sum_{|S|=k} w(N-S)$. It is unnecessary to check $k < \frac{n}{2}$ since $\frac{n}{n/k} w(N) > 2w(N)$, while the other side is less than $2w'(N)$. So it is clear that $E_{N,v} < 1$. On the other hand when $k > \frac{n}{2}$, $\frac{n}{n/k} w(N) < 2w(N)$. It is enough to check the extreme $k = n-1$.

$$\frac{n}{n-1} w(N) = w'(N) + \frac{1}{n-1} \sum_{i=1}^{n} w(i) < w'(N) + \frac{1}{n-1} w(N) \qquad \text{(because the game is essential)}$$

Thus $w(N) < w'(N)$ or $E_{N,v} < 1$.

Next consider $E_{N,v} > 0.5$. It is sufficient to consider $w'(N) = \frac{1}{\binom{n-1}{k-1}} \sum_{|S|=k} w(S)$. It is clear that $k \geqslant \frac{n}{2}$ (because of superadditivity $\sum_{\substack{|S|=\ell \\ \ell < \frac{n}{2}}} w(S) \leqslant \sum_{|S|=n-\ell} w(N-S)$ since in both we have the same number of elements in the sum). Next we know (again because of superadditivity) that $\sum_{|S|=k} w(S) < \binom{n}{k} w(N)$.

Thus,

$$w'(N) = \frac{1}{\binom{n-1}{k-1}} \sum_{|S|=k} w(S) < \frac{\binom{n}{k}}{\binom{n-1}{k-1}} w(N) = \frac{n}{k} w(N).$$

Hence,

$\frac{w(N)}{w'(N)} = E_{N,v} > \frac{k}{n}$. But the least value for k is $\frac{n}{2}$ and then $E_{N,v} > 0.5$.

Q.E.D.

5. A Test of Homocore Predictions

A rare opportunity to experientially test the homocore as a predictor of human divisional agreements was afforded by the data collected at the University of Wisconsin by Michener et al [1982]. (This follows several important evaluative papers, [8, 9, 10, 11].) Six different 5-person characteristic function games were repeatedly "solved" by 180 students acting as players. (The games are defined in Table 3 of their paper.) The actual imputations reached were compared to the predictions of various solution concepts (the Shapley value, the Nucleolus, the 2-Center, the Disruption value) each applied to a set of modifications of the characteristic function. A discrepenacy measure was defined as follows:

$$\underline{d} = \left[\sum_{i=1}^{n} (x_i - p_i)^2 \right]^{1/2} ,$$ where x_i is the actual payoff to player i

and p_i is the predicted payoff. Michener et al concluded from the discrepancies presented in their Table 6 that the homomollifier (of Charnes, Rousseau and Seiford [4]) was superior to the other modifications of the characteristic function in describing the "worth" of each coalition. This independently supports our idea of calculating the homocore from the homomollifier and not directly from the characteristic function.

Applying the homocore as a predictor and comparing it with the other predictions by the addition of one line to Table 6 of [7], we obtain the following results:

Mean Discrepancy Scores as a Function of Representation and Solution Concept

Solution Concept	Characteristic Function v(S)	Counter Homomollifier k(S)	Equal Mollifier e(S)	Homomollifier h(S)	Complement $\overline{v}(S)$
Shapley value	17.25	19.90	17.25	15.92	17.25
Nucleolus	24.76	20.42	16.11	15.81	16.09
2-Center	19.71	21.94	19.71	17.97	19.71
Disruption value	16.49	19.74	17.78	16.49	19.74
Homocore	--	--	--	11.58	--
Average over Solution	19.55	20.50	17.71	16.55*	18.20

*This average does not include the homocore score.

Clearly none of the other predictions come close to the discrepancy score achieved by the homocore. It is 1.365 times smaller than its closest competitor (also derived from the homomollifier). This table of results also suggests that the complicated threat, counter-threat, counter-counter threat, etc. regressions involved in solution notions such as the bargaining set or the nucleolus may not be good representations of the factors in real behavior for reaching divisional agreements. But, both for the latter conclusion and for the conclusion of superiority of the homocore, much more experimentation is evidently required.

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Defects in the core as a solution to n-person superadditive characteristic function games are examined and the process of achieving a "reasonable" core by changing the value of the grand coalition is studied. A new unique solution concept, the "homocore," is proposed based on the "homomollifier" notion interpreted as the result of implicit bargaining and weighted averages of coalitional worth. Thereby unreasonable cores are excluded but a core-like dominance property on the average is maintained.

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20. ABSTRACT (continued)

It also yields a measure of "operational efficiency" where the game may be interpreted as an economic situation of decreasing marginal productivity.

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